Expansions of the Ising Model Spin-Spin Correlation

Allan Trinh with supervisor Nicholas Witte

BACKGROUND

In theoretical physics, the two-dimensional Ising model is one the simplest statistical mechanical system whose exact solution show a phase transition. Interestingly the diagonal correlation $<\sigma_{0,0}\sigma_{N,N}>$ has the Toeplitz determinant form

$$< \sigma_{0,0}\sigma_{N,N} > = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

Where the elements are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{k\cos(n\theta) - \cos[(n-1)\theta]}{\sqrt{k^2 + 1 - 2k\cos\theta}}\, d\theta$$

In a paper by Jimbo and Miwa in 1980, this diagonal correlation also satisfy the following Painleve VI equation.

$$[t(t-1)\frac{d^2\sigma}{dt^2}]^2 = N^2[(t-1)\frac{d\sigma}{dt} - \sigma]^2 - 4\frac{d\sigma}{dt}[(t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}](t\frac{d\sigma}{dt} - \sigma)$$

Where for $T > T_c$ ($t := k^2$)

$$\sigma(t) = t(t-1)\frac{d}{dt}\log < \sigma_{0,0}\sigma_{N,N} > - \frac{1}{4}$$

And for $T < T_c$ ($t := k^{-2}$)

$$\sigma(t) = t(t-1)\frac{d}{dt}\log < \sigma_{0,0}\sigma_{N,N} > - \frac{t}{4}$$

The behavior of the diagonal correlation is very well understood for low and high temperature regimes. But what happens at around the phase transition temperature T_c for a given parameter N?

MY PROJECT

My project was to find the exact solution to this second order non-linear differential equation when T -> T_c. Using established information about the Toeplitz elements for both low and high temperature regimes, they can be generated as a linear combination of complete elliptic integrals of the first and second kind from a very simple recurrence relation. (This can be checked through many different methods)

Applying the asymptotic behavior of the complete elliptic integrals, the solution of the Toeplitz determinant can be rewritten in the form:

$$< \sigma_{0,0}\sigma_{N,N} > = \sum_{m=0}^{\infty}\sum_{n=0}^{N} D_{m,n}(1-t)^m[\log(1-t)]^n$$

Substituting this form into the Painleve differential equation, you can extract a recurrence relation for $D_{m,n}$. Doing this for both the low and high temperature regimes and assuming the continuity of the σ-function, the picture for the diagonal correlation near the transition temperature becomes complete.

MY EXPERIENCE

The vacation research program was a very insightful and rewarding experience. You certainly come across new and interesting mathematics. Even though I do not have a complete solution to this problem yet, the process in getting there was a terrific learning experience simply because the challenges it offers cannot be easily found in the normal classroom.

I would like to give a big thank you to my supervisor, Nicholas Witte. I was very new to this topic but Nicholas took his time going through necessary background material with me and I appreciate that. Also a thank you to the maths department for facilitating a wonderful working environment and providing great support.